Yau 2025 Applied Math – Individual Contest

1. Consider the following boundary value problem for the stationary Schrödinger equation:

$$-\Delta u(\mathbf{x}) + V(\mathbf{x})u(\mathbf{x}) = 0 \quad \text{in } \Omega; \quad \partial_\nu u(\mathbf{x}) = \psi(\mathbf{x}) \quad \text{on } \partial\Omega, \tag{1}$$

   where $V \in L^2(\Omega)$, ν denotes the exterior unit normal vector to $\partial\Omega$, and $\psi \in H^{-1/2}(\partial\Omega)$. It is assumed that 0 is not a Neumann eigenvalue of the operator $-\Delta + V$ in Ω, ensuring the boundary value problem (1) is well-posed with a unique solution $u \in H^1(\Omega)$ for any ψ. For fixed ψ, define the boundary dataset:

$$\mathcal{M}_\psi := u|_{\partial\Omega} \in H^{1/2}(\partial\Omega), \tag{2}$$

   where u solves (1). The inverse problem is to recover V from $\mathcal{M}_\psi$ with a given ψ, formulated as the operator equation:

$$F(V) = \mathcal{M}_\psi, \tag{3}$$

   where F is defined via (1).

   (a) Prove that F in (3) is Fréchet differentiable and derive its derivative.

   (b) Suppose $V \in \mathrm{Span}\{V_1(\mathbf{x}), \ldots, V_N(\mathbf{x})\}$ for given functions V_j. Sketch a Newton-type method for solving (3) using (a).

2. Consider the linear transport equation with variable coefficients:

$$u_t + a(x)u_x + a'(x)u = 0, \quad x \in [0,1],\ t > 0,$$

   with periodic boundary conditions and initial data $u(x,0) = u_0(x)$. Assume $a(x) \in C^2([0,1])$, and $a(x) > 0$ for all x.

   (a) Rewrite the PDE in conservation form and explain the meaning of conservation in this context.

   (b) Construct a second-order central difference scheme using numerical fluxes $F^n_{j+1/2}$. Ensure that your scheme is at least conditionally stable.

   (c) Derive a CFL condition for the case $a(x) \equiv a_0$ (constant) by applying von Neumann stability analysis.

3. Consider a smooth simple closed curve $\mathcal{C}$ with a bounded interior region Ω in $\mathbb{R}^2$. Formulate a variational problem to show that, among all such curves with a fixed enclosed area A, the circle minimizes the perimeter L.

4. In $\mathbb{R}^3$, prove that the shape of the intersection of

   - the unit 4-norm ball $x^4 + y^4 + z^4 \leq 1$, and
   - the plane $x + y + z = 0$,

   is a disk. (This is a simplified problem that arises in conic mathematical optimization.)